Advanced Studies in Pure Mathematics 63, 2012
Galois–Teichmüller Theory and Arithmetic Geometry
pp. 813–832

Some congruence properties of Eisenstein invariants associated to elliptic curves

Hiroaki Nakamura

§1. Introduction

Let π be a free profinite group with free generators $\mathbf{x}_1, \mathbf{x}_2$ and let π' (resp. π'') denote the commutator (resp. double-commutator) subgroup of π. Regard the full automorphism group $\mathsf{A} := \mathrm{Aut}(\pi)$ acting on the left of π. The purpose of this paper is to study some elementary arithmetic properties of a certain series of invariants

$$\mathbb{E}_m : \mathsf{A} \times \hat{\mathbb{Z}}^2 \longrightarrow \hat{\mathbb{Z}} \quad (m \in \mathbb{N})$$

reflecting the action of A on the meta-abelian quotient π/π''. In particular, we shall introduce a canonical series of finite index subgroups of A fully exhausting congruity of the invariants $\mathbb{E}_m$ in a systematical way.

Motivation to this paper came from our previous work [N10] where π was given as the fundamental group of an affine elliptic curve $E : y^2 = 4x^3 - g_2 x - g_3$ over a field K of characteristic zero. A choice of a K-rational tangential base point at infinity of the elliptic curve E gives rise to a natural Galois representation $\varphi : \mathrm{Gal}(\bar{K}/K) \to \mathsf{A}$. Given π being presented as $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{z} \mid [\mathbf{x}_1, \mathbf{x}_2]\mathbf{z} = 1 \rangle$ so that $\mathbf{z}$ generates an inertia over the infinity puncture, we introduced in loc. cit. certain arithmetic invariants

$$\mathbb{E}_m : \mathrm{Gal}(\bar{K}/K) \times \hat{\mathbb{Z}}^2 \longrightarrow \hat{\mathbb{Z}} \quad (m \in \mathbb{N})$$

(induced from φ) that converge to the "Eisenstein measure" $\mathcal{E}_\sigma$ ($\sigma \in \mathrm{Gal}(\bar{K}/K(E_{tor}))$) of [N95]–[N99]. Especially, we showed an explicit formula for $\mathbb{E}_m$ in terms of Kummer properties of modular units evaluated at E. By Galois correspondence, those finite index subgroups of A obtained in this paper yield a sequence of finite Galois extensions of K that

Received May 30, 2011.
Revised February 8, 2012.
This work was partially supported by JSPS KAKENHI 21340009.