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Comparison of some quotients of fundamental groups of algebraic curves over p-adic fields

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§0. Introduction

Basic problems related to lifting and reduction of etale covers of curves had been treated in the fundamental work of Grothendieck [4], which established unique liftability, as well as good reduction property for Galois covers with degrees not divisible by the residue characteristic p. This applies also to tame covers, say, of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, in which case Raynaud [16] proved a partial but yet unsurpassed result for Galois covers of degree divisible by p but not by p^2. Historically, there is another line of investigations started mainly by Shimura and Igusa. In [5], Igusa made a basic contribution to the case of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ by geometric method, proving that the modular tower of levels not divisible by p (the Galois degrees can be divisible by p) has good reduction. The theory of Shimura curves [19, 20] provided extremely rich arithmetic systems of curves and source of further studies. In connection with fundamental groups, we just recall here the following; each tower obtained by reduction of modular or Shimura curves can be characterized, inside the tower of curves with prescribed tame ramifications, only by the complete splitting of "special $\mathbf{F}_{q^2}$-rational points" [6, 8, 9, 10]. As for developments after 1980's related to the study of the algebraic fundamental groups of curves, we shall leave their descriptions to other articles of this Volume.

Now, here, we take up the following question. Let $p > 2$, let $\bar{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p = \mathbf{Z}/p$, and $R_0 = \mathbb{W}[[\bar{\mathbf{F}}_p]]$ be the ring of Witt vectors. Note that R_0 does not contain the group μ_p of p-th roots of unity. Let k_0 be the quotient field of R_0, and G_{k_0} be the absolute Galois group $G_{k_0} = \mathrm{Gal}(\bar{k}_0/k_0)$. Let X be a proper smooth R_0-scheme whose fibers $X_\eta = X \otimes k_0$ and $X_s = X \otimes \bar{\mathbf{F}}_p$ are geometrically irreducible curves. Pick an R_0-section $x = (x_\eta, x_s) \in X(R_0)$. Then G_{k_0} acts on the

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