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On the double zeta values

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§0. Introduction

In a very important recent paper [1], F. Brown solved long standing conjectures about multiple zeta values (here abbreviated as MZV). In particular, he showed that any such series

$$(1) \qquad \zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}$$

(with integers $n_1 \geq 1, \ldots, n_{r-1} \geq 1, n_r \geq 2$) can be expressed as *a linear combination with rational coefficients of special values* $\zeta(m_1, \ldots, m_s)$ *where each* m_i *is* 2 *or* 3. The uniqueness of such a linear combination is beyond reach for the moment, but F. Brown [1], after A. B. Goncharov [2] has promoted the MZV's to *motivic multizeta values* $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$, and shown that the $\zeta^{\mathfrak{m}}(m_1, \ldots, m_s)$'s with m_i in $\{2, 3\}$ form a rational basis of the space of the motivic MZV's.

In the course of his proof, he needs an identity of the form

$$(2) \qquad H(a,b) = \sum_{i=1}^{k} \alpha_i^{a,b} \, \zeta(2i+1) \, H(k-i)$$

with $k = a + b + 1$ and[1]

$$(3) \qquad H(m) := \zeta(\underbrace{2, \ldots, 2}_{m}), \quad H(a,b) := \zeta(\underbrace{2, \ldots, 2}_{a}, 3, \underbrace{2, \ldots, 2}_{b}).$$

F. Brown was not able to give an explicit formula for the rational coefficients $\alpha_i^{a,b}$, but this was supplied by D. Zagier [5], thus completing the proof by F. Brown. It is known since Euler that, for a given integer $m \geq 1$, the numbers $H(m)/\pi^{2m}$, $\zeta(2m)/\pi^{2m}$ and $\zeta(2)^m/\pi^{2m}$ are all rational, and $\zeta(0) = -\frac{1}{2}$. So in the statement of formula (2), one

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[1]We use the convention $H(0) = 1$.