Advanced Studies in Pure Mathematics 56, 2009
Singularities — Niigata–Toyama 2007
pp. 321–340

Čech–Dolbeault cohomology and the $\bar{\partial}$-Thom class

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The Čech–de Rham cohomology theory for C^∞ manifolds was initiated by A. Weil [17] and is fully explained in [5]. Combined with the Chern–Weil theory of characteristic classes, this cohomology theory, especially its relative version, is particularly suited to describe the localization theory of characteristic classes related to Chern polynomials. The techniques are effective even on some singular varieties (e.g., [11], [12], [14], [16]). We may express the Poincaré, Alexander and Thom homomorphisms in terms of this cohomology as well, see [6] for a combinatorial definition of these homomorphisms. Note also that the Thom class of a vector bundle can be naturally defined in this cohomology and, in the case of complex vector bundles, it coincides with the localization of the top Chern class of the pull-back bundle with respect to the diagonal section [14]. This point of view helps us very much in clarifying the local behavior of a section near its zeros.

In this article, we discuss complex analytic analogues of the above, replacing the Čech–de Rham cohomology by the "Čech–Dolbeault cohomology" and the Chern classes by the classes introduced by M. Atiyah in [2]. To be a little more specific, we first develop a theory of Čech–Dolbeault cohomology, its relative version and related topics in Sections 1, 2 and 3. In Section 4, we introduce the notion of "$\bar{\partial}$-integration along the fiber" for later use. We recall, in Sections 5 and 6, the Atiyah classes for holomorphic vector bundles and the localization theory of these classes, which is exploited in detail in [1]. In Section 7, we define the "$\bar{\partial}$-Thom class" of a holomorphic vector bundle as the localization of the top Atiyah class of the pull-back bundle with respect to the diagonal section. We then prove the "$\bar{\partial}$-Thom isomorphism", and in this situation, we have a satisfactory expression of the "$\bar{\partial}$-Alexander homomorphism". Finally, in Section 8, we indicate a way of getting a good description of the $\bar{\partial}$-Alexander homomorphism in a more general setting.

Received May 20, 2008.
Revised September 17, 2008.
Partially supported by a grant of JSPS.