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A conjectural presentation of fusion algebras

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Dedicated to Prof. Masaki Kashiwara on his sixtieth birthday

§1. Introduction

Let G be a connected, simply-connected, simple algebraic group over $\mathbb{C}$. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. We denote their Lie algebras by $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ respectively. Let $P_+ \subset \mathfrak{h}^*$ be the set of dominant integral weights. For any $\lambda \in P_+$, let $V(\lambda)$ be the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ. We fix a positive integer ℓ and let $\mathcal{R}_\ell(\mathfrak{g})$ be the free $\mathbb{Z}$-module with basis $\{V(\lambda) : \lambda \in P_\ell\}$, where

$$P_\ell := \{\lambda \in P_+ : \lambda(\theta^\vee) \leq \ell\},$$

θ is the highest root of $\mathfrak{g}$ and $\theta^\vee$ is the associated coroot. There is a product structure on $\mathcal{R}_\ell(\mathfrak{g})$, called the *fusion product* (cf. Section 3), making it a commutative associative (unital) ring. In this paper, we consider its complexification $\mathcal{R}_\ell^\mathbb{C}(\mathfrak{g})$, called the *fusion algebra*, which is a finite dimensional (commutative and associative) algebra without nilpotents.

Let $\mathcal{R}(\mathfrak{g})$ be the Grothendieck ring of finite dimensional representations of $\mathfrak{g}$ and let $\mathcal{R}^\mathbb{C}(\mathfrak{g})$ be its complexification. As given in 3.6, there is a surjective ring homomorphism $\beta : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$. Let $\beta^\mathbb{C} : \mathcal{R}^\mathbb{C}(\mathfrak{g}) \to \mathcal{R}_\ell^\mathbb{C}(\mathfrak{g})$ be its complexification and let $I_\ell(\mathfrak{g})$ denote the kernel of $\beta^\mathbb{C}$. Since $\mathcal{R}_\ell^\mathbb{C}(\mathfrak{g})$ is an algebra without nilpotents, $I_\ell(\mathfrak{g})$ is a radical ideal.

The main aim of this note is to conjecturally describe this ideal $I_\ell(\mathfrak{g})$. Before we describe our result and conjecture, we briefly describe the known results in this direction. Identify the complexified representation ring $\mathcal{R}^\mathbb{C}(\mathfrak{g})$ with the polynomial ring $\mathbb{C}[\chi_1, \ldots, \chi_r]$, where r is the rank of $\mathfrak{g}$ and χ_i denotes the character of $V(\omega_i)$, the i^{th} fundamental representation of $\mathfrak{g}$. It is generally believed (initiated by the physicists) that

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