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A-integrability of geodesic flows and geodesic equivalence

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§1. Introduction

A common property of the metrics that admit non-trivial geodesic equivalence is that their geodesic flows admit families of pairwise commuting integrals of a special type (see [14, 7, 16, 17, 13]). The metrics studied in [1] and [20] have integrals with analogous properties. We consider this property as a definition of a new class of (pseudo) Riemannian metrics that we call *A-integrable metrics.* We prove that these metrics inherit the main properties of the metrics that admits non-trivial geodesic equivalence (see [16]).

Let M^n be a smooth n-dimensional manifold. By $\Gamma(E)$ we denote the space of the smooth sections of the vector bundle $\pi : E \longrightarrow M^n$. Let $P \in \Gamma(\mathrm{Hom}(TM, TM))$. We say that P is *diagonalizable over* $\mathbb{R}$ *on* M^n if for every point $x \in M^n$ there exists a (real) basis in T_xM^n such that the operator $P(x)$ has diagonal form $P(x) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Definition 1. *Let g be a (pseudo) Riemannian metric on the manifold M^n and let $A \in \Gamma(\mathrm{Hom}(TM, TM))$ be a self-adjoint operator which is diagonalizable over $\mathbb{R}$ on M^n with respect to g. The metric g is called* A-integrable *if the functions from the one-parameter family*

$$I_c(\xi) \stackrel{\mathrm{def}}{=} \det(A + c\mathbf{1})g((A + c\mathbf{1})^{-1}\xi, \xi) \tag{1}$$

are in involution with respect to the symplectic structure $\omega_g \stackrel{\mathrm{def}}{=} FL_g^*\omega$. Here $FL_g : TM \to T^*M$ is the Legendre transformation corresponding to the metric g and ω is the canonical symplectic structure on T^*M.

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