Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 401–488

Geometries and symmetries of soliton equations and integrable elliptic equations

Chuu-Lian Terng

§1. Introduction

In the classical literature, a differential equation is called "integrable" if it can be solved by quadratures. A Hamiltonian system in $2n$-dimensions is *completely integrable* if it has n independent commuting Hamiltonians. By the Arnold-Liouville Theorem, such systems have action-angle variables that linearize the flow, and these can be found by quadrature. This concept of integrability can be extended to PDEs, and one class consists of evolution equations on function spaces that have Hamiltonian structures and are completely integrable Hamiltonian systems in the sense of Liouville, i.e., there exist action angle variables. We call this class of equations soliton equations. The model examples are the Korteweg-de Vries equation, the non-linear Schrödinger equation (NLS equation), and the Sine-Gordon equation (SGE equation). For example, the action-angle variables are constructed for the KdV equation in [33], for the NLS equation in [34], and for flows in the $SL(n)$-hierarchy in [5]. Besides the Hamiltonian formulation and complete integrability, these soliton equations have many other remarkable properties including:

(1) infinite families of explicit solutions,

(2) a hierarchy of commuting flows described by partial differential equations,

---

Received March 18, 2002.
Revised January 24, 2003.
2000 *Mathematics Subject Classification.* Primary 37K25; Secondary 53A, 53C.
Research supported in part by NSF Grant DMS 9972172 and National Center for Theoretical Sciences, Taiwan.