Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 315–337

Isoparametric geometry and related fields

Reiko Miyaoka

§1. Introduction

The purpose of this paper is to give a perspective in the theory of isoparametric hypersurfaces and related topics. Starting with a brief introduction of the subject, we explain what is now going on, describing important results and remaining problems as well as new aspects.

The classification problem is now being solved in the cases $g = 4, 6$, where g is the number of principal curvatures. In §3, we give a simple proof of Cartan's theorem on the classification for $g = 3$, and generalize the strategy to any g. For this we use the relation between the curvature and the Lax equation, as well as Singer's strongly curvature-homogeneous theorem. On the other hand, the behavior of the kernel of the differential of the Gauss map of the focal submanifolds is important. In §4, we treat the degeneracy of the Gauss map, which seems related to the homogeneity.

Some isoparametric hypersurfaces and all the focal submanifolds provide us with many examples of austere submanifolds, whose twisted normal cones are special Lagrangian submanifolds in $\mathbb{C}^n$. In §5, we introduce this topic. We explain how to obtain explicit solutions of the special Lagrangian equation from isoparametric functions (§5.3). Special Lagrangian submanifolds are volume minimizing, and the topology of stable minimal submanifolds is restricted. We discuss the topology of austere submanifolds in §6, using Morse theory as in the proof of Lefschetz' theorem. In §7, we discuss hypersurface geometry from the viewpoint of Hamiltonian systems of hydrodynamic type.

Received February 8, 2002.
Revised March 3, 2007 (updated).
2000 *Mathematics Subject Classification.* Primary 53C42.