Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 285–314

Harmonic tori and their spectral data

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One of the earliest applications of modern integrable systems theory (or "soliton theory") to differential geometry was the solution of the problem of finding all constant mean curvature (CMC) tori in $\mathbb{R}^3$ (and therefore, by taking the Gauss map, finding all non-conformal harmonic maps from a torus to S^2). At its simplest level this proceeds from the recognition that the Gauss-Codazzi equations of a CMC torus are the elliptic sinh-Gordon equations

$$u_{z\bar{z}} + \sinh(4u) = 0, \;\; z = x + iy. \tag{1}$$

It was shown in the late 1980's ([24, 1]) that each doubly periodic solution of this equation can be written down in terms of the Riemann θ-function for a compact Riemann surface X, called the spectral curve (this also follows from Hitchin's work [10] on harmonic tori in S^3, which used a distinctly different approach). That this is true relies on two observations. First, (1) has a zero-curvature (or Lax pair) representation: it is the condition that

$$[\frac{\partial}{\partial z} - U_\zeta, \frac{\partial}{\partial \bar{z}} + U^\dagger_{\bar{\zeta}^{-1}}] = 0, \; U_\zeta = \begin{pmatrix} u_z & e^{-2u}\zeta^{-1} \\ e^{2u}\zeta^{-1} & -u_z \end{pmatrix}, \; \forall \zeta \in \mathbb{C}^*,$$

where '†' denotes the Hermitian transpose. As a result this equation belongs to a hierarchy of infinitely many commuting equations, so that solutions to (1) may belong to an infinite dimensional family of deformations through solutions. These deformations are called the "higher flows" of the sinh-Gordon hierarchy. Secondly, each independent higher flow contributes to the number of independent Jacobi fields which the CMC surface admits: these belong to the kernel of the elliptic operator

Received January 30, 2001.
Revised September 3, 2001.
2000 *Mathematics Subject Classification.* Primary 53C43.
Partially supported by EPSRC Grant GR/M97534.