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A generalized Weierstrass representation for a submanifold S in $\mathbb{E}^n$ arising from the submanifold Dirac operator

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§1. Introduction

Using the submanifold quantum mechanical scheme [dC, JK], the restricted Dirac operator for a k dimensional spin (k-spin) submanifold S immersed in Euclidean space $\mathbb{E}^n$ $(0 < k < n)$ was defined [BJ, Ma1-10]. We call it the submanifold Dirac operator. The zero modes of the Dirac operator express the local properties of the submanifold, such as the Frenet-Serret and generalized Weierstrass formulae. We shall give a survey of this method from the point of view of quantum physics.

As motivation, we recall three facts.

(1) Let us consider an element Q of a ring of operators $\mathbf{P}$ defined over a Riemannian manifold M. The concept of the adjoint of Q is very subtle, as we shall explain briefly, following the book of Björk (see [Remark 1.2.16 in Bj]). Assume that M is Riemannian. For smooth functions f_1 and f_2 whose support is compact, we consider the following integral as a bilinear form of f_1 and f_2:

$$\int_M d\,\mathrm{vol}\ (f_1 Q f_2). \tag{1-1}$$

What is the natural adjoint of Q? One might regard the action on f_1 obtained by integration by parts as defining the adjoint. However the measure here depends on the local coordinates.

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