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Quantum product, topological recursion relations, and the Virasoro conjecture

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The Virasoro conjecture predicts that the generating function of the Gromov-Witten invariants is annihilated by infinitely many differential operators which form a half branch of the Virasoro algebra. This conjecture was proposed by Eguchi, Hori and Xiong [EHX] and S. Katz (cf. [CK] [EJX]). It is a natural generalization of a conjecture of Witten (cf. [W2] [Ko] [W2]) and provides a powerful tool in the computation of Gromov-Witten invariants. The genus-0 Virasoro conjecture was proved in [LT] (cf. [DZ] and [G3] for alternative proofs). The genus-1 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved in [DZ]. Without assuming semisimplicity, the genus-1 Virasoro conjecture was reduced to the genus-1 L_1-constraint on the small phase space in [L1]. It was also proved in [L1] that the genus-1 Virasoro conjecture holds if the quantum cohomology is not too degenerate (a condition weaker than semisimplicity). The essential part of the results in [L1] was extended to the genus-2 Virasoro conjecture in [L2]. The study of the genus-2 Virasoro conjecture is important because this is the first case where we do not have a formula to reduce the problem to the small phase space. The behavior of the Virasoro conjecture in this case will provide much needed insight in what we should expect in the higher genera cases. The techniques developed in [L2] could be easily adapted to the study of the higher genera Virasoro conjecture.

In this expository article, we will explain how to apply the main ideas in [L2] to the study of the Virasoro conjecture in all genera. In particular, we will explain how to use the quantum product on the big

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