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Special Lagrangian 3-folds and integrable systems

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§1. Introduction

This is the sixth in a series of papers [17, 18, 19, 20, 21] constructing explicit examples of special Lagrangian submanifolds (SL m-folds) in $\mathbb{C}^m$. The principal motivation for the series is to study the singularities of SL m-folds, especially when $m = 3$. This paper also has a second objective, which is to connect SL m-folds with the theory of integrable systems, and to arouse interest in special Lagrangian geometry within the integrable systems community.

We begin in §2 with a brief introduction to *special Lagrangian submanifolds* in $\mathbb{C}^m$, which are a class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, defined using calibrated geometry. Section 3 then gives a rather longer introduction to *harmonic maps* $\psi : S \to \mathbb{CP}^{m-1}$, where S is a Riemann surface. Such maps form an *integrable system*, and have a complex and highly-developed theory involving the Toda lattice equations, loop groups, and classification using spectral curves.

Section 4 explains the connection of this with special Lagrangian geometry. Let N be a *special Lagrangian cone* in $\mathbb{C}^3$, and set $\Sigma = N \cap \mathcal{S}^5$. Then Σ is a *minimal Legendrian surface* in $\mathcal{S}^5$, and so the image of a *conformal harmonic map* $\phi : S \to \mathcal{S}^5$ from a Riemann surface S. The projection $\psi = \pi \circ \phi$ of ϕ from $\mathcal{S}^5$ to $\mathbb{CP}^2$ is also conformal and harmonic, with Lagrangian image.

Thus, ψ can be analyzed in the integrable systems framework of §3. As the image of ψ is Lagrangian there is a simplification, in which the SU(3) Toda lattice equation reduces to the *Tzitzéica equation*, and the spectral curve acquires an extra symmetry. We use the integrable

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