Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 163–187

From CMC surfaces to Hamiltonian stationary Lagrangian surfaces

Frédéric Hélein and Pascal Romon

§1. Introduction

Minimal surfaces and surfaces with constant mean curvature (CMC) have fascinated differential geometers for over two centuries. Indeed these surfaces are solutions to variational problems whose formulation is elegant, modelling physical situations involving soap films and bubbles; however their richness has not been exhausted yet. Advances in the understanding of these surfaces draw on complex analysis, theory of Riemann surfaces, topology, nonlinear elliptic PDE theory and geometric measure theory. Furthermore, one of the most spectacular developments in the past twenty years has been the discovery that many problems in differential geometry – including those of minimal and CMC surfaces – are actually integrable systems.

The theory of integrable systems developed in the 1960's, beginning essentially with the study of a now famous example: the Korteweg-de Vries equation, $u_t + 6uu_x + u_{xxx} = 0$, modelling waves in a shallow flat channel[1]. In the 1960's mathematicians noticed the exceptional properties of the KdV equation: existence of solitary waves that "superpose" almost linearly, and an infinite number of conserved quantities. From these observations, C. Gardner, J. Greene, M. Kruskal and R. Miura [12] showed in 1967 that this equation could be solved completely by

Received December 27, 2000.
Revised February 8, 2002.
2000 *Mathematics Subject Classification.* Primary 49Q05; Secondary 53C42.

[1]Earlier already, in 1955, Fermi, Pasta and Ulam had unexpectedly discovered the soliton phenomenon (to their great surprise) while simulating a one-dimensional model in statistical mechanics on the Los Alamos computer.