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Generalized Weierstraß representations of surfaces

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§1. Introduction

The classical Weierstraß representation

$$\phi(z,\bar{z}) = \mathrm{Re}\left\{\int_{z_0}^{z} (\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg)dz\right\}$$

has been a very useful tool for the construction and the investigation of minimal surfaces in $\mathbb{R}^3$. While the differential equation describing these surfaces is highly nonlinear, the Weierstraß data, a pair consisting of a holomorphic function f and a meromorphic function g, is completely unconstrained. Moreover, the relation between the Weierstraß data (f,g) and ϕ is sufficiently direct that it is possible to relate geometric properties of the surface to the properties of the Weierstraß data.

In recent years a generalized Weierstraß representation was found, which applies to the construction of all surfaces of constant mean curvature in $\mathbb{R}^3$. If the mean curvature H vanishes, i.e. if the surface actually is a minimal surface, then the new procedure leads to the classical Weierstraß representation in a straightforward fashion. If $H \neq 0$, then the generalized Weierstraß representation is a new tool for the construction of conformal immersions of these surfaces. The Weierstraß data consists again of a pair of functions (Q,f), where Q is holomorphic and f is meromorphic. More precisely, Qdz^2 is the Hopf differential of the surface to be constructed and f is closely related with the conformal factor of the induced metric. In spite of this, the relation between the Weierstraß data and the geometry of the conformally immersed surface is much less direct than in the classical case. However, it turns out that at least some

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