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Exploring surfaces through methods from the theory of integrable systems: The Bonnet problem

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A generic surface in Euclidean 3-space is determined uniquely by its metric and curvature. Classification of all special surfaces where this is not the case, i.e. of surfaces possessing isometries which preserve the mean curvature, is known as the Bonnet problem. Regarding the Bonnet problem, we show how analytic methods of the theory of integrable systems — such as finite-gap integration, isomonodromic deformation, and loop group description — can be applied for studying global properties of special surfaces.

§1. Quaternionic description of surfaces. Bonnet problem

1.1. Differential equations of surfaces

Let $\mathcal{F}$ be a smooth orientable surface in 3-dimensional Euclidean space. The Euclidean metric induces a metric Ω on this surface, which in turn generates the complex structure of a Riemann surface $\mathcal{R}$. Under such a parametrization, which is called *conformal*, the surface $\mathcal{F}$ is given by an immersion

$$F = (F_1, F_2, F_3) : \mathcal{R} \to \mathbb{R}^3,$$

and the metric is conformal: $\Omega = e^u\, dz d\bar{z}$, where z is a local coordinate on $\mathcal{R}$.

One should keep in mind that a complex coordinate is defined up to holomorphic $z \to w(z)$ transformation. This freedom will be used to simplify the corresponding equations.

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