Advanced Studies in Pure Mathematics 45, 2006
Moduli Spaces and Arithmetic Geometry (Kyoto, 2004)
pp. 43–73

On integral Hodge classes on uniruled or Calabi-Yau threefolds

Claire Voisin

To Masaki Maruyama, on his 60th birthday

§0. Introduction

Let X be a smooth complex projective variety of dimension n. The Hodge conjecture is then true for rational Hodge classes of degree $2n-2$, that is, the degree $2n-2$ rational cohomology classes of X which are of Hodge type $(n-1, n-1)$ are algebraic, which means that they are the cohomology classes of algebraic cycles with $\mathbb{Q}$-coefficients. Indeed, this follows from the hard Lefschetz theorem, which provides an isomorphism:

$$\cup c_1(L)^{n-2} : H^2(X, \mathbb{Q}) \cong H^{2n-2}(X, \mathbb{Q}),$$

from the fact that the isomorphism above sends the space of rational Hodge classes of degree 2 onto the space of rational Hodge classes of degree $2n-2$, and from the Lefschetz theorem on $(1,1)$-classes.

For integral Hodge classes, Kollár [11], (see also [14]) gave examples of smooth complex projective manifolds which do not satisfy the Hodge conjecture for integral degree $2n-2$ Hodge classes, for any $n \geq 3$. The examples are smooth general hypersurfaces X of certain degrees in $\mathbb{P}^{n+1}$. By the Lefschetz restriction theorem, such a variety satisfies

$$H^2(X, \mathbb{Z}) = \mathbb{Z}H, \ H = c_1(\mathcal{O}_X(1)),$$

and

$$H^{2n-2}(X, \mathbb{Z}) = \mathbb{Z}\alpha, \ < \alpha, H >= 1.$$

Plane sections C of X have cohomology class $[C] = d\alpha$, $d = deg\, X$, because

$$deg\, C = d =< [C], H > .$$

Kollár [11] proves the following :

Received December 14, 2004.