Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 127–133

Recent development on Grauert domains

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§1. Introduction

The purpose of this article is to give a short survey on the recent development of a canonical complex structure, the so called *adapted complex structure*, on the tangent bundle of a real-analytic Riemannian manifold.

It was observed by Grauert [G] that a real-analytic manifold X could be embedded in a complex manifold as a maximal totally real submanifold. One way to see this is to complexify the transition functions defining X. However, this complexification is not unique. In [G-S] and [L-S], Guillemin-Stenzel and independently Lempert-Szöke encompass certain conditions on the ambient complex structure to make the complexification canonical for a given real-analytic Riemannian manifold. In short, they were looking for a complex structure, on part of the cotangent bundle T^*X, compatible with the canonical symplectic structure on T^*X. Equivalently, it is to say that there is a unique complex structure, the *adapted complex structure*, on part of the tangent bundle of X making the leaves of the Riemannian foliation on TX into holomorphic curves. The set of tangent vectors of length less than r equipped with the adapted complex structure is called a *Grauert tube* T^rX. For each X, there corresponds a $r_{max}(X) \geq 0$ which is the maximal real number such that the adapted complex structure is defined on T^rX for all $r \leq r_{max}(X)$. Though each Grauert tube over the same Riemannian manifold are diffeomorphic to each other, it was proved in [K1] and [Sz1] that T^rX and T^sX are biholomorphically nonequivalent when $r \neq s$. A domain D in which the adapted complex structure is defined and $X \subset D \subset TX$, is called a *Grauert domain*. The largest one of such Grauert domains is called the *maximal Grauert domain* in TX. In general, the maximal Grauert domain is strictly larger than $T^{r_{max}}X$. They are the same when X is a symmetric space of rank-one. The domain of definition depends on the geometry of X. Lempert and Szöke have the following estimate on the existence of domain of definition.

Received March 25, 2002