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Complex Analysis in Several Variables
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Short $\mathbb{C}^k$

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§1. Introduction

One of Oka's main contributions was to solve the Levi problem.

There are various ways to generalize the Levi Problem. The Union problem is one: Let $\Omega_0 \subset \Omega_1 \subset \cdots \subset \cup\Omega_n = \Omega$. Suppose that each Ω_j is Stein. Is Ω Stein? To approach the Union Problem, one can try at first to understand the simplest cases of Ω.

Example 1.1. *Long* $\mathbb{C}^2$. *Suppose that each* Ω_j *is biholomorphic to* $\mathbb{C}^2$. *Then we call* Ω *a long* $\mathbb{C}^2$. *It is an open question whether all long* $\mathbb{C}^2$ *are actually biholomorphic to* $\mathbb{C}^2$.

Example 1.2. (*Fornæss*, ([*F*, 1976])) *In dimension* 3 *and higher it can happen that* Ω *is not Stein and that each* Ω_n *is biholomorphic to a ball.*

This left open the question in dimension 2.

Theorem 1.3. (*Fornæss-Sibony*, ([*FS*, 1981])) *Suppose that each* Ω_j *is biholomorphic to the unit ball in* $\mathbb{C}^2$. *If the (infinitesimal) Kobayashi metric of* Ω *is not identically zero, then* Ω *is biholomorphic to the ball or to* $\Delta \times \mathbb{C}$, *where* Δ *is the unit disc.*

Recall that the (infinitesimal) Kobayashi metric of Ω vanishes identically if and only if for all $p \in \Omega$ and any tangent vector ξ to Ω at p and for any $R > 0$, there exists a holomorphic map $f : \Delta = \{z \in \mathbb{C}; |z| < 1\} \to \Omega$ so that $f(0) = p$ and $f'(0) = R\xi$.

This theorem left still open the case when the Kobayashi metric vanishes identically. The most obvious example of such a case is when $\Omega = \mathbb{C}^2$. However, the question remaining was whether there was any

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