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Cubic Schrödinger: The Petit Canonical Ensemble

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§1. Introduction

This report describes some aspects of the Gibbsian petit canonical ensemble for the cubic Schrödinger equation in the space of functions of period 1, say. §2–5 (defocussing case) represent joint work with K. Vaninsky[1]. §6 is a brief report on the much more difficult focussing case. The original hope, that the petit ensemble might provide a picture of the typical solution, is far from being achieved.

1.1. Preliminaries[2]

The mechanical state is a pair QP of nice functions of period 1, moving according to the defocussing flow:

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -\frac{\partial^2 P}{\partial x^2} + (Q^2+P^2)\,P &= \frac{\partial H_3}{\partial P} \\ \frac{\partial P}{\partial t} &= +\frac{\partial^2 Q}{\partial x^2} - (Q^2+P^2)\,Q &= -\frac{\partial H_3}{\partial Q} \end{aligned}$$

This is a Hamiltonian system, relative to the classical bracket in function space, with Hamiltonian

$$H_3 = \frac{1}{2}\int_0^1 \left[(Q')^2 + (P')^2\right] + \frac{1}{4}\int_0^1 (Q^2+P^2)\,.$$

It is integrable in the full technical sense of the word, having an infinite series of (commuting) constants of motion $H_1 = \frac{1}{2}\int_0^1 (Q^2+P^2)$, $H_2 = \int_0^1 Q'P$, H_3, and so on. The flow is integrated with the help of the Dirac equation

$$M' = \left[\begin{pmatrix} Q & P \\ P & -Q \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] M$$

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1) McKean-Vaninsky [1997]

2) Manakov et al. [1984] and/or McKean-Vaninsky [1997]