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Equivariant Diffusions on Principal Bundles

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Let $\pi : P \longrightarrow M$ be a smooth principal bundle with structure group G. This means that there is a C^∞ right multiplication $P \times G \longrightarrow P$, $u \mapsto u \cdot g$ say, of the Lie group G such that π identifies the space of orbits of G with the manifold M and π is locally trivial in the sense that each point of M has an open neighbourhood U with a diffeomorphism

$$\begin{array}{ccc} \tau_U : \pi^{-1}(U) & \longrightarrow & U \times G \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

over U, which is equivariant with respect to the right action of G, i.e. if $\tau_u(b) = (\pi(b), k)$ then $\tau_u(b \cdot g) = (\pi(b), kg)$. Assume for simplicity that M is compact. Set $n = dimM$. The fibres, $\pi^{-1}(x)$, $x \in M$ are diffeomorphic to G and their tangent spaces $VT_uP(= kerT_u\pi)$, $u \in P$, are the 'vertical' tangent spaces to P. A *connection* on P, (or on π) assigns a complementary 'horizontal' subspace HT_uP to VT_uP in T_uP for each u, giving a smooth horizontal sub-bundle HTP of the tangent bundle TP to P. Given such a connection it is a classical result that for any C^1 curve: $\sigma : [0, T] \longrightarrow M$ and $u_0 \in \pi^{-1}(\sigma(0))$ there is a unique horizontal $\tilde{\sigma} : [0, T] \longrightarrow P$ which is a lift of σ, i.e. $\pi(\tilde{\sigma}(t)) = \sigma(t)$ and has $\tilde{\sigma}(0) = u_0$.

In his startling ICM article [8] Itô showed how this construction could be extended to give horizontal lifts of the sample paths of diffusion processes. In fact he was particularly concerned with the case when M is given a Riemannian metric $\langle , \rangle_x$, $x \in M$, the diffusion is Brownian motion on M, and P is the orthonormal frame bundle $\pi : OM \longrightarrow M$. Recall that each $u \in OM$ with $u \in \pi^{-1}(x)$ can be considered as an isometry $u : \mathbb{R}^n \longrightarrow T_xM$, $\langle , \rangle_x$ and a

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