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Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions

Masatoshi Noumi and Yasuhiko Yamada

§ Introduction

This paper is an outcome of our attempt to understand internal connections among several appearances of the subtraction-free birational transformations.

There is a well-known procedure for passing from subtraction-free rational functions to piecewise linear functions. Roughly, this is the procedure of replacing the operations

$$ab \to a+b, \quad a/b \to a-b, \quad a+b \to \max\{a,b\} \quad (\text{or } \min\{a,b\}).$$

It can be applied consistently to an arbitrary rational function expressed as a ratio of two polynomials with positive real coefficients, in order to produce a combination of $+$, $-$ and max (or min), representing a piecewise linear function. In combinatorics, this procedure has been employed for the *algebraization* of combinatorial algorithms. A large class of combinatorial algorithms can be described as piecewise linear transformations among discrete variables which take integer values. For such a piecewise linear transformation, it is meaningful in many cases to find a good subtraction-free rational counterpart; algebraic computation of subtraction-free rational functions may possibly bring out unexpected solutions to combinatorial problems. For this *tropical approach* to combinatorics, we refer the reader to [1], [14] and the references therein.

In the context of discrete integrable systems, the same procedure is known as *ultra-discretization* [27]. A remarkable example is the ultra-discretization of discrete Toda equation which provides with soliton cellular automata, called the box-ball systems [28]. It is already recognized that the theory of box-ball systems is precisely the dynamics of

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