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An induction theorem for Springer's representations

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§1. Statement of the result

1.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic p. We fix a prime number l different from p. Let G be a connected reductive algebraic group over $\mathbf{k}$. Let W be the Weyl group of G. Let $\mathcal{B}$ be the variety of Borel subgroups of G. For any $g \in G$ let $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$. According to Springer [S], W acts naturally on the l-adic cohomology $H^n(\mathcal{B}_g)$. (Springer's original definition of the W action is valid only when g is unipotent and p is 0 or is large. Here we adopt the definition given in [L1] which is valid without restrictions on g and p.)

1.2. Let L be a Levi subgroup of a parabolic subgroup P of G. Let W' be the Weyl group of L (naturally a subgroup of W). Let $\mathcal{B}'$ be the variety of Borel subgroups of L (naturally a subvariety of $\mathcal{B}$). Let $u \in L$ be unipotent. Let $\mathcal{B}'_u = \{B' \in \mathcal{B}'; u \in B'\}$. Then the W-module $H^n(\mathcal{B}_u)$ and the W'-module $H^n(\mathcal{B}'_u)$ are well defined.

Theorem 1.3. *We have*

$$\sum_n (-1)^n H^n(\mathcal{B}_u) = \mathrm{ind}_{W'}^{W}\Big(\sum_n (-1)^n H^n(\mathcal{B}'_u)\Big)$$

(*equality of virtual W-modules*).

This result was stated without proof in [AL] in the case where $p = 0$. Here we provide a proof valid for any p (answering a question that J. C. Jantzen asked me).

In the remainder of this paper we assume that $p > 1$ and that $\mathbf{k}$ is an algebraic closure of the finite field F_p with p elements. (By standard results, if the theorem holds for such $\mathbf{k}$ then it holds for any $\mathbf{k}$.)

Let $\mathcal{Z}$ be the identity component of the centre of L. Clearly, the theorem is a consequence of Propositions 1.4, 1.5 below (these will be proved in Sections 2 and 3 respectively).

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