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Operator means and their norms

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§1. Introduction

In his very interesting (unpublished) 1979 notes [17] A. McIntosh obtained the following arithmetic-geometric mean inequality for Hilbert space operators H, K, X:

$$||HXK|| \leq \frac{1}{2}||H^*HX + XKK^*||. \tag{1}$$

Among other things he also pointed out that simple alternative proofs for so-called Heinz-type inequalities ([9], and see also the discussions in §2) are possible based on this inequality. Then, about 15 years later Bhatia and Davis ([4]) noticed that the inequality remains valid for all unitarily invariant norms (including the Schatten norms $||\cdot||_p$ and so on). Recall that a norm $|||\cdot|||$ for Hilbert space operators is called unitarily invariant when $|||UXV||| = |||X|||$ for unitary operators U, V, and basic facts on these norms can be found for example in [8, 10, 19]. In recent years the arithmetic-geometric mean and related inequalities have been under active investigation by several authors, and very readable accounts on this subject can be found in [1, 3].

Motivated by all of the above, the authors have investigated simple unified proofs for known (as well as some new) norm inequalities, some refinement of the norm inequality (1) (such as the arithmetic-logarithmic-geometric mean inequality), and a general theory on operator (and/or matrix) means in a series of recent articles [15, 11, 12]. The purpose of the present notes is to give a brief survey on the topics dealt in these articles.

We will derive a variety of integral expressions for relevant operators to establish desired norm inequalities. This means that our arguments are not just for proving norm inequalities, but we are actually solving

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