Global and Local Properties of Pencils of Algebraic Curves

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§0. Introduction

Let S be a non-singular projective surface over $\mathbb{C}$, and let $f : S \to B$ be a relatively minimal fibration of curves of genus g over a non-singular projective curve B of genus b. In this article, we discuss some recent developments in the area where its global and local properties interact each other, with comments on several interesting open questions.

From the global point of view, our motivation comes from the study of minimal surfaces of general type. In the birational sense, any algebraic surface has a fibration over a curve, because it has an algebraic function. For surfaces of small Kodaira dimension, we can choose among various pencils a "preassigned" pencil such as Mori fibrations or Iitaka fibrations. On the other hand, there seems to be no canonical way in finding a pencil which reflects well the structure of a surface of general type. However, we often see that a pencil structure appears naturally for them as well. In the series of papers [42], Horikawa showed that most surfaces which are geographically close to the Noether line, $K_S^2 = 2\chi(\mathcal{O}_S) - 6$, have a pencil of curves of genus 2 which is induced on S from a ruling of its canonical image via the canonical map. Similar phenomena can be observed for canonical surfaces close to the Castelnuovo line $K_S^2 = 3\chi(\mathcal{O}_S) - 10$ ([22], [10]). In this case, the quadric hull of the canonical image is a threefold of small degree and its ruling usually induces on S a pencil of non-hyperelliptic curves of genus 3 (see [52], [44]). One can find a lot of such observations in literatures (e.g., Xiao's works) indicating the importance of a systematic study of fibered surfaces in the study of surfaces of general type.