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$|\mathbf{Hom}(A, G)|$ (III)

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§1. Introduction

For a finite group G, its *Frobenius number* h_n^{cyc} is the number of solutions of the equation $x^n = 1$ in G and a *Sylow number* s_n^{cyc} is the number of cyclic subgroups of G of order n. These numbers are named after Frobenius theorem and Sylow's theorem ([Yo 96]). The classical Frobenius theorem states that h_n^{cyc} is divisible by the greatest common divisor of n and $|G|$. The following *transition formula* holds:

$$h_n^{\mathrm{cyc}} = \sum_{r|n} \varphi(r) s_r^{\mathrm{cyc}}, \quad (n \geq 1), \tag{1}$$

where φ denotes the Euler function.

Now define the *zeta functions of Sylow and Frobenius types* by

$$\begin{aligned} S_G^{\mathrm{cyc}}(z) &:= \sum_{n=1}^{\infty} \frac{\varphi(n) s_n^{\mathrm{cyc}}}{n^z} = \sum_{g \in G} |g|^{-z}, \\ H_G^{\mathrm{cyc}}(z) &:= \sum_{n=1}^{\infty} \frac{h_n^{\mathrm{cyc}}}{n^z}. \end{aligned}$$

Then the transition formula can be presented by the *transition identity* between these functions as follows:

$$H_G^{\mathrm{cyc}}(z) = \zeta(z) S_G^{\mathrm{cyc}}(z), \tag{2}$$

where the transition function $\zeta(z)$ is Riemann's zeta function. Another expression of the transition formula (1) is given by the following cyclotomic identity:

$$\prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n} \right)^{\sharp\{g \in G \mid |g| = n\}/n} = \exp\left(\sum_{n=1}^{\infty} \frac{h_n^{\mathrm{cyc}}}{n} t^n \right). \tag{3}$$

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