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The Essentials of Monstrous Moonshine

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This is a fast introduction to Monstrous Moonshine.

All our functions expanded at $\tau = i\infty$ have the form:

$$(*) \qquad f(\tau) = \frac{1}{q} + \sum_{k \geq 0} a_k q^k, \quad q = e^{2i\pi\tau}, \quad \Im(\tau) > 0, \quad a_k \in \mathbb{C}.$$

We further assume that $a_0 = 0$ (standard form) for convenience, and that $a_k \in \mathbb{Q}$ (to ensure trivial Galois action). For replicable functions there is a reasonable conjecture that the a_k are algebraic integers - this, too, we assume. We find that the coefficients of classical modular functions known to Jacobi, Fricke, and Klein, are related to the characters of $\mathbb{M}$, the Monster simple sporadic group, in that, to each conjugacy class of cyclic subgroups $\langle g \rangle$, of $\mathbb{M}$, there is such a function, j_g with coefficient of $q^k = \text{Trace}(H_k(g))$ for some representation, H_k, (the k^{th} Head representation) of $\mathbb{M}$.

In November 1978 I wrote to John Thompson that $196884 = 1 + 196883$, relating the coefficient of q in the elliptic modular function, $j(\tau)$, to the degree of the smallest faithful complex representation of $\mathbb{M}$. Little was then known to me of the degrees of irreducible characters of $\mathbb{M}$ but I did have access to those of $E_8(\mathbb{C})$ and related an initial sequence of them to the q-coefficients of the cube root of j. This was quickly disposed of by Victor Kac [Kac], see also [Lep].

There are 194 conjugacy classes of $\mathbb{M}$, 172 classes of cyclic subgroups, and 171 distinct functions j_g. This, and more, is to be found in Conway-Norton [CN]. All these functions are genus zero in that this is the genus of the compactified Riemann surface $\widehat{G_f \backslash \mathcal{H}}$ where G_f is the discrete invariance group of f, acting on the upper half-plane, $\mathcal{H}$.

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