Advanced Studies in Pure Mathematics 32, 2001
Groups and Combinatorics — in memory of Michio Suzuki
pp. 291–300

Generation Theorems for Finite Groups

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§1. Introduction

This article is a survey of the author's work on generation theorems for finite groups. The starting point is:

Theorem A (J. G. Thompson 1968). *A finite group is soluble if and only if every two elements generate a soluble subgroup.*

Thompson obtained this result as a corollary of his classification of the minimal simple groups [12]. A direct proof has been obtained by the author [3]. A natural question to ask is:

what happens if we keep one of the generators fixed?

For a finite group G we define

$$\mathrm{sol}(G)$$

to be the largest normal soluble subgroup of G.

Conjecture B. *Let x be an element of the finite group G. Then*

$$x \in \mathrm{sol}(G) \quad \text{if and only if} \quad \langle x, y\rangle \quad \text{is soluble for all} \quad y \in G.$$

The author has not yet been able to prove this conjecture. However, much progress has been made and will be described in what follows.

In order to illustrate one of obstacles to proving Conjecture B, we present a small but crucial part of the author's proof of Theorem A. Henceforth, the word *group* will mean *finite group.*

Lemma 1.1 (D. Goldschmidt [2]). *Let z be a p-element of the soluble group H. Then*

$$O_{p'}(C_H(z)) \leq O_{p'}(H).$$

Received June 1, 1999.
Revised April 20, 2000.