Advanced Studies in Pure Mathematics 32, 2001
Groups and Combinatorics — in memory of Michio Suzuki
pp. 255–277

Steiner systems and Mathieu groups revisited

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These notes about an old topic of Witt (Abh. Hbg. 1938) describe a further approach to the relevant existence and isomorphism theorems for Steiner systems. Some standard information about the automorphism groups is obtained along the way. Actually, I wish to proceed by group theoretic arguments as much as possible. Besides Sylow's Theorem, including

$$|G : N_G(P)| \equiv 1(p) \quad \text{for} \quad P \in Syl_p(G),$$

and the most obvious properties of the 2-dimensional linear groups over $GF(11)$ and $GF(9)$, they mainly require some formalities around transitive action of a group G on a set Ω, above all

$$|\Omega| = |G : G_\alpha| \quad \text{for} \quad \alpha \in \Omega.$$

I also mention the "Frattini argument", "Witt's Lemma", and the concept of a Frobenius group:

The first gives $G = HG_\alpha = G_\alpha H$ for any transitive subgroup H, the second states that the normalizer $N_G(X)$ of a subgroup (or subset) $X \subseteq G_\alpha$ is transitive on the set Ω_X of fixed points if (and only if) X is "very weakly closed" in G_α, that is all G-conjugates $X^g \subseteq G_\alpha$ are already conjugate to X in G_α. The standard X besides $X = G_\alpha$ is a Sylow subgroup of G_α. Trivially, Witt's Lemma implies the analogous result for n-fold transitive groups.

Thirdly, to say that G is a Frobenius group on Ω, means that $1 \neq G_\alpha \neq G$ and $G_{\alpha\beta} = 1$ for all $\beta \neq \alpha$. We ignore Frobenius' famous theorem and assume also that G_α has a complement K in G. Then K is regular on Ω, is the set of all elements of G not conjugate to an element $\neq 1$ of G_α, and is called the Frobenius kernel of G. Accordingly, an abstract Frobenius group is a semi-direct product $G = KA$ (with K normal) such that the above holds for a suitable "G-set" Ω and with $A = G_\alpha$, or equivalently no element of K commutes with an element

Received September 22, 1999.