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Representation Theory in Characteristic p

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Let k be an algebraically closed field of characteristic p. (Thus, p is either 0 or a prime number.) Let G be a group which is at the same time an affine algebraic variety over k (that is, an algebraic group over k). A *representation* of G is a homomorphism $\rho : G \longrightarrow GL(V)$ of G into the group of automorphisms of a finite dimensional k-vector space V which is at the same time a morphism of algebraic varieties. We also say that V is a *G-module.* We say that V is *irreducible* if $V \neq 0$ and there is no subspace V' of V (other than 0 or V) such that $\rho(g)V' \subset V'$ for all $g \in G$.

Let $\mathfrak{g}$ be the Lie algebra of G. A *representation* of $\mathfrak{g}$ is a k-linear map $\tau : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ where V is a finite dimensional k-vector space such that

$$\tau([\xi,\xi']) = \tau(\xi)\tau(\xi') - \tau(\xi')\tau(\xi)$$

for all $\xi, \xi' \in \mathfrak{g}$. We also say that V is a *$\mathfrak{g}$-module.* The notion of irreducibility of a $\mathfrak{g}$-module is defined in the same way as in the group case.

We will assume that G is connected, almost simple (that is, G has finite centre and G modulo its centre is a simple group) and simply connected (in a suitable sense). Chevalley [C] proved the remarkable result that the classification of such G is the same as the classification of simple complex Lie algebras (achieved by É. Cartan and Killing). Thus, G must be a special linear group, a symplectic group, a spin group or one of five exceptional groups.

The problem that we will discuss in this paper is that of classifying the irreducible G-modules and $\mathfrak{g}$-modules and that of understanding as much as possible the structure of those irreducible modules. Work on these problems have occupied mathematicians throughout much of this century. We will review some of this work. Towards the end of the paper we will engage in speculation about possible future directions.

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