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# Approximation of Expectation of Diffusion Process and Mathematical Finance

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## §1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{(B^1(t), \dots, B^d(t)\ ;\ t \in [0,\infty)\}$ be a $d$-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose devivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that with probability one $X(t,x)$ is continuous in $t$ and smooth in $x$.

In many fields, it is important to compute $E[f(X(T,x))]$ numerically, where $f$ is a function defined in $\mathbf{R}^N$. Let $u(t,x) = E[f(X(t,x))]$, $t > 0$, $x \in \mathbf{R}^N$. Then $u$ satisfies the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x), \\ u(0,x) = f(x). \end{cases}$$

Here $L = \frac{1}{2}\sum_{i=1}^N V_i^2 + V_0$. So to compute $E[f(X(T,x))]$ is the same to compute the solution $u(T,x)$ to PDE. However, in mathematical finance, if we think of the problem of pricing of Europian options, there are sometimes following difficulties.

(1) $L$ can be degenerate. Moreover, $L$ may not satisfy even the Hörmander condition.

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