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Stably Free and Not Free Rings of Integers

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Let $N/\mathbb{Q}$ be a tame Galois extension of number fields with finite degree $[N : \mathbb{Q}]$ and Galois group G. One knows by the normal basis theorem that N is a free rank one $\mathbb{Q}[G]$-module. It is natural to look at the structure of the ring of integers $\mathcal{O}_N$ as a $\mathbb{Z}[G]$-module. The first known result is Hilbert's theorem which asserts that *If G is abelian and the discriminant of $N/\mathbb{Q}$ is prime to $[N : \mathbb{Q}]$ then $\mathcal{O}_N$ has a normal integral basis* ([Hi] satz 132); that is to say, there exists an algebraic integer $a \in N$ such that $\mathcal{O}_N$ has a basis made of the set $\{g(a) \mid g \in G\}$. The following result is E. Nœther's theorem which asserts that $\mathcal{O}_N$ is $\mathbb{Z}[G]$-projective if and only if $N/\mathbb{Q}$ is a tame extension; in fact Nœther's result shows that $\mathcal{O}_N$ is locally-free: for all prime p the extended module $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_N$ is $\mathbb{Z}_p[G]$-free with rank one. We can associate to $\mathcal{O}_N$ its image $[\mathcal{O}_N]$ in the projective class group $\mathrm{Cl}(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$-modules; from now on, all the extensions will be tame. In 1968, J. Martinet proved that if G is a dihedral group of order $2p$, p an odd prime then $\mathcal{O}_N$ is $\mathbb{Z}[G]$-free. A few years after, in the case where $G = H_8$ (the quaternionic group of order 8), he was able to describe $\mathrm{Cl}(\mathbb{Z}[G]) \simeq \{\pm 1\}$ and to give a criterion for $\mathcal{O}_N$ free or not. Moreover, he produced rings of integers $\mathbb{Z}[G]$-free and not free ([Ma2]). Almost in the same time, Armitage gave examples of L-functions of quaternionic fields with a zero at $s = \frac{1}{2}$ ([A]).

Knowing the two results J-P. Serre did computations in [S] on examples and was surprised to see that the constant of the functional equation of the Artin L-series for the irreducible degree two character of $\mathrm{Gal}(N/\mathbb{Q})$ was 1 whenever $\mathcal{O}_N$ was free and -1 in the other cases. This was proved to be a theorem by A. Fröhlich [F1].

The following years A. Fröhlich proposed a nice conjecture finally established by M.J. Taylor [T]. We give a few notations before we state this theorem.

The first step was a description of the projective classgroup as a quotient of a group of equivariant maps from R_G (the group of characters

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