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# Class Numbers of Imaginary Quadratic Fields

**Winfried Kohnen**

## §1. Introduction

Starting with Gauss, class numbers of quadratic fields always have been very interesting and quite mysterious objects. Here we would like to survey some more recent results concerning indivisibility of class numbers of imaginary quadratic fields by prime numbers. For more details we refer the reader to [1].

In the following, we denote by $D < 0$ the discriminant of an imaginary quadratic field. We let $h(D)$ be the class number, i.e. the order of the class group $CL(D)$ of $\mathbf{Q}(\sqrt{D})$ .

We want to study the question "how often" $h(D)$ is not divisible by a given prime number $l$.

## §2. Classical results

**Theorem** (Gauss). *Let $t$ be the number of different prime divisors of $D$. Then $h(D)$ is odd if and only if $t = 1$.*

In fact, by genus theory one has $CL(D)\,/CL(D)^2 \cong (\mathbf{Z}\,/2\mathbf{Z})^{t-1}$. Now use the structure theorem for finite abelian groups.

**Theorem** (Hartung, 1974). *Let $l$ be an odd prime. Then there exist infinitely many $D < 0$ such that $h(D) \not\equiv 0 \pmod{l}$.*

Let us sketch the *proof.* For a natural number $N$ with $N \equiv 0, 3 \pmod 4$ let $H(N)$ be the Hurwitz-Kronecker class number, i. e. the class number of positive definite binary quadratic forms of discriminant $-N$ where each class $C$ is counted with multiplicity $\frac{1}{\sharp Aut(C)}$. If $-N = Df^2$ with $f \in \mathbf{N}$ and $D$ a fundamental discriminant, then

$$H(N) = \frac{h(D)}{w(D)} \sum_{d|f} \mu(d)(\frac{D}{d})\sigma_1(\frac{f}{d}),$$

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