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Recent Progress on the Finiteness of Torsion Algebraic Cycles

Noriyuki Otsubo

In this article we review recent results on the finiteness of torsion algebraic cycles on certain surfaces over number fields.

1. Let X be an algebraic variety over a field k. For an integer $i \geq 0$, let X^i be the set of schematic points of codimension i (equivalently, the set of integral closed subvarieties of codimension i of X). The Chow group of algebraic cycles of codimension d modulo rational equivalence is defined by

$$CH^d(X) := \operatorname{Coker}\left(\bigoplus_{x \in X^{d-1}} \kappa(x)^* \stackrel{\text{div}}{\to} \bigoplus_{x \in X^d} \mathbb{Z}\right)$$

where $\kappa(x)$ denotes the residue field at x. Chow groups are natural generalization of Picard group, but little is known on their structure in general.

Bloch was the first to study the close relation between algebraic cycles and algebraic K-theory. Let $K_n(X)$ be the algebraic K-group defined by Quillen. Let $\mathcal{K}_n$ be the Zariski sheaf on X which is associated to the presheaf $U \mapsto K_n(\Gamma(U, \mathcal{O}_X))$. Define similarly $\mathcal{H}^n(\mathbb{Z}/l^m(r))$ by the sheafification of the étale cohomology functor $U \mapsto H^n_{\text{et}}(U, \mu_{l^m}^{\otimes r})$ for a prime number $l \neq \text{ch}(k)$. Then, if X is smooth we have the following isomorphisms called Bloch's formula:

$$CH^d(X) \simeq H^d_{\text{Zar}}(X, \mathcal{K}_d),$$
$$CH^d(X)/l^m \simeq H^d_{\text{Zar}}(X, \mathcal{H}^d(\mathbb{Z}/l^m(d))).$$

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