Advanced Studies in Pure Mathematics 30, 2001
Class Field Theory – Its Centenary and Prospect
pp. 79–86

Embedding Problems with restricted Ramifications and the Class Number of Hilbert Class Fields

Akito Nomura

§1. Introduction

Let k be an algebraic number field of finite degree, and $\mathfrak{G}$ its absolute Galois group. Let L/k be a finite Galois extension with Galois group G, and $(\varepsilon) : 1 \to A \to E \xrightarrow{j} G \to 1$ a group extension with an abelian kernel A. Then an embedding problem $(L/k, \varepsilon)$ is defined by the diagram

$$(*) \qquad \begin{array}{ccccccccc} & & & & & & \mathfrak{G} & & \\ & & & & & & \downarrow{\scriptstyle \varphi} & & \\ (\varepsilon) : 1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{j} & G & \longrightarrow & 1 \end{array}$$

where φ is the canonical surjection. When (ε) is a central extension, we call $(L/k, \varepsilon)$ a central embedding problem. A solution of the embedding problem $(L/k, \varepsilon)$ is, by definition, a continuous homomorphism ψ of $\mathfrak{G}$ to E satisfying the conditions $j \circ \psi = \varphi$. We say the embedding problem $(L/k, \varepsilon)$ is solvable if it has a solution. The Galois extension over k corresponding to the kernel of any solution is called a solution field. A solution ψ is called a proper solution if it is surjective. The existence of a proper solution of $(L/k, \varepsilon)$ is equivalent to the existence of a Galois extension $M/L/k$ such that the canonical sequence $1 \to \mathrm{Gal}(M/L) \to \mathrm{Gal}(M/k) \to \mathrm{Gal}(L/k) \to 1$ coincides with ε.

Let S be a finite set of primes of L. An embedding problem with ramification conditions $(L/k, \varepsilon, S)$ is defined by the diagram $(*)$, which is same to the case of $(L/k, \varepsilon)$. A solution ψ is called a solution of $(L/k, \varepsilon, S)$ if M/L is unramified outside S, where M is the solution field corresponding to ψ. We remark that these definitions are a little different from those in [3] and [8], but essentially of the same nature.

Received August 28, 1998.
Revised October 24, 1998.