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The Milnor fiber as a virtual motive

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In this text, which correponds to our talk at the Conference "Singularities in Geometry and Topology" held in Sapporo in July 1998, we present our results, obtained in collaboration with Jan Denef, on the virtual motive associated to the Milnor fiber.

§1. Introduction

1.1. Let X be a smooth and connected complex algebraic variety and consider $f : X \to \mathbf{C}$ a non constant morphism. For any singular point x of $f^{-1}(0)$, the Milnor fiber at x is defined as

$$F_x := B(x, \varepsilon) \cap f^{-1}(t),$$

for $0 < |t| \ll \varepsilon \ll 1$, with $B(x, \varepsilon)$ the open ball of radius ε centered at x. There is some abuse of notation here, since, strictly speaking, F_x depends on the choice of ε and t, but all the invariants we shall consider will not.

Maybe the most natural invariants of the Milnor fiber to look at first are the Betti numbers

$$b_i(F_x) := \mathrm{rk} H^i(F_x, \mathbf{C}).$$

In fact, these numbers are in general very difficult to compute as soon as the singularity of $f = 0$ at x is not isolated. Much more easy to determine is the Euler characteristic

$$\chi(F_x) := \sum_i (-1)^i b_i(F_x).$$

When X is of dimension n and the singularity of $f = 0$ at x is isolated, $\chi(F_x) = 1 + (-1)^{n-1} b_{n-1}(F_x)$, and $b_{n-1}(F_x)$ is nothing else but the Milnor number.

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