Advanced Studies in Pure Mathematics 28, 2000
Combinatorial Methods in Representation Theory
pp. 261–285

Capelli Elements in the Classical Universal Enveloping Algebras

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For any complex classical group $G = O_N, Sp_N$ consider the ring $\mathrm{Z}(\mathfrak{g})$ of G-invariants in the corresponding enveloping algebra $\mathrm{U}(\mathfrak{g})$. Let u be a complex parameter. For each $n = 0, 1, 2, \ldots$ and every partition ν of n into at most N parts we define a certain rational function $Z_\nu(u)$ which takes values in $\mathrm{Z}(\mathfrak{g})$. Our definition is motivated by the works of Cherednik and Sklyanin on the reflection equation, and also by the classical Capelli identity. The degrees in $\mathrm{U}(\mathfrak{g})$ of the values of $Z_\nu(u)$ do not exceed n. We describe the images of these values in the n-th symmetric power of $\mathfrak{g}$. Our description involves the plethysm coefficients as studied by Littlewood, see Theorem 3.4 and Corollary 3.6.

§1. Capelli elements in the algebra $\mathrm{U}(\mathfrak{gl}_N)$

We work with the general linear Lie algebra $\mathfrak{gl}_N$ over the complex field $\mathbb{C}$. In this section we recall the definition from [OO1, S] of the Capelli elements in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. Here we also recall an explicit construction from [N2,O] of these elements.

Let the indices i, j run through the set $\{1, \ldots, N\}$. Let the vectors e_i form the standard basis in $\mathbb{C}^N$. We fix in the Lie algebra $\mathfrak{gl}_N$ the basis of the standard matrix units E_{ij}. We will also regard E_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. Now choose the Borel subalgebra in $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqslant j$. Then choose the basis $E_{11}, \ldots, E_{NN}$ in the corresponding Cartan subalgebra.

Let ν be any partition of n into at most N parts. We will write $\nu = (\nu_1, \ldots, \nu_N)$. Let U_ν be the irreducible $\mathfrak{gl}_N$-module of highest weight ν. The module U_ν appears in the decomposition of the n-th tensor power of the defining $\mathfrak{gl}_N$-module $\mathbb{C}^N$. It is called the *polynomial* $\mathfrak{gl}_N$-module corresponding to the partition ν.

There is a distinguished basis in the centre $\mathrm{Z}(\mathfrak{gl}_N)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$, parametrized by the same partitions ν. The

Received November 23, 1998.