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Cohomology of Local systems

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§1. Introduction

This survey is intended to provide a background for the authors paper [23]. The latter was the subject of the talk given by the second author at the Arrangement Workshop. The central theme of this survey is the cohomology of local systems on quasi-projective varieties, especially on the complements to algebraic curves and arrangements of lines in $\mathbf{P}^2$. A few of the results of [23] are discussed in section 4 while the first part of this paper contains some of highlights of Deligne's theory [7] and several examples from the theory of Alexander invariants developed mostly by the first author in the series of papers [17] - [22]. We also included several problems indicating possible further development. The second author uses the opportunity to thank M. Oka and H. Terao for the hard labor of organizing the Arrangement Workshop.

§2. Background on cohomology of local systems

Local systems. A local system of rank n on a topological space X is a homomorphism $\pi_1(X) \to GL(n, \mathbf{C})$. Such a homomorphism defines a vector bundle on X with discrete structure group or a locally constant bundle (cf. [7], I.1). Indeed, if $\tilde{X}_u$ is the universal cover of X then $\tilde{X}_u \times_{\pi_1(X)} \mathbf{C}^n$ is such a bundle (this product is the quotient of $\tilde{X}_u \times \mathbf{C}^n$ by the equivalence relation $(x, v) \sim (x', v')$ if and only if there is $g \in \pi_1(X)$ such that $x' = gx, v' = gv$; this quotient has the projection onto $\tilde{X}_u/\pi_1(X) = X$ with the fiber $\mathbf{C}^n$). Vice versa, any locally constant bundle defines a representation of the fundamental group of the base.

If X is a *complex* manifold, then there is a one-to-one correspondence between the local systems and pairs consisting of a *holomorphic* vector bundle on X and an *integrable* connection on the latter (cf. [7] I.2, Theorem 2.17). If V is a vector bundle then a connection can be viewed as a $\mathbf{C}$-linear map defined for each open set U of X and acting as follows:

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