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Vassiliev Invariants of Braids and Iterated Integrals

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§ Introduction

The notion of finite type invariants of knots was introduced by Vassiliev in his study of the discriminats of function spaces (see [13]). It was shown by Kontsevich [9] that such invariants, which we shall call the Vassiliev invariants, can be expressed universally by iterated integrals of logarithmic forms on the configuration space of distinct points in the complex plane.

In the present paper we focus on the Vassiliev invariants of braids. Our main object is to clarify the relation between the Vassiliev invariants of braids and the iterated integrals of logarithmic forms on the configuration space which are homotopy invariant. A version of such description for pure braids is given in [6]. We denote by B_n the braid group on n strings. Let J be the ideal of the group ring $\mathbf{C}[B_n]$ generated by $\sigma_i - \sigma_i^{-1}$, where $\{\sigma_i\}_{1 \leq i \leq n-1}$ is the set of standard generators of B_n. The vector space of the Vassiliev invariants of B_n of order k with values in $\mathbf{C}$ can be identified with $\mathrm{Hom}(\mathbf{C}[B_n]/J^{k+1}, \mathbf{C})$. Let us stress that such vector space had been studied in terms of the iterated integrals due to K. T. Chen before the work of Vassiliev. We introduce a graded algebra $\tilde{\mathcal{A}}_n$, which is a semi-direct product of the completed universal enveloping algebra of the holonomy Lie algebra of the configuration space and the group algebra of the symmetric group. We construct a homomorphism $\theta : B_n \longrightarrow \tilde{\mathcal{A}}_n$ expressed as an infinite sum of Chen's iterated integrals, which gives a universal expression of the holonomy of logarithmic connections. This homomorphism may be considered as a prototype of the Kontsevich integral for knots. Using this homomorpshim we shall determine all iterated integrals of logarithmic forms which provide invariants of braids (see Theorem 3.1). As a Corollary we recover the isomorphism

$$\tilde{\mathcal{A}}_n \cong \varprojlim \mathbf{C}[B_n]/J^j$$