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On $\wedge\mathfrak{g}$ for a Semisimple Lie Algebra $\mathfrak{g}$, as an Equivariant Module over the Symmetric Algebra $S(\mathfrak{g})$

Bertram Kostant

§1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathcal{C}$ be the set of all commutative Lie subalgebras $\mathfrak{a}$ of $\mathfrak{g}$. If $\mathfrak{a} \in \mathcal{C}$ and $k = dim\, \mathfrak{a}$ let $[\mathfrak{a}] = \wedge^k \mathfrak{a}$. Regard $[\mathfrak{a}]$ as a 1-dimensional subspace of $\wedge^k \mathfrak{g}$ and let $C \subset \wedge \mathfrak{g}$ be the span of all $[\mathfrak{a}]$ for all $\mathfrak{a} \in \mathcal{C}$. The exterior algebra $\wedge\mathfrak{g}$ is a $\mathfrak{g}$-module with respect to the extension, θ, of the adjoint representation, defined so that $\theta(x)$ is a derivation for any $x \in \mathfrak{g}$. It is obvious that $C = \sum_{k=1}^{n} C^k$ is a graded $\mathfrak{g}$-submodule of $\wedge\mathfrak{g}$. Of course $C^k = 0$ for $k > n_{abel}$ where n_{abel} is the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$. The paper [4] initiated a study of the $\mathfrak{g}$-module C. It was motivated by a result of Malcev giving the value of n_{abel} for all complex simple Lie subalgebras. For example, for the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8, the value of n_{abel}, respectively, is $3, 9, 16, 27$ and 36. See [10].

One of the results in [4] is that C (denoted by A in [4]) is a multiplicity free $\mathfrak{g}$-module. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. If Ξ is an index set for the set of all abelian ideals $\{\mathfrak{a}_\xi\}$, $\xi \in \Xi$, of $\mathfrak{b}$, then the irreducible components of C may also be indexed by Ξ. The irreducible components, written as C_ξ, $\xi \in \Xi$, are characterized by the property that $[\mathfrak{a}_\xi]$ is the highest weight space of C_ξ. One therefore has the unique decomposition

$$C = \sum_{\xi \in \Xi} C_\xi$$

into irreducible components. Sometime after [4] was published, Dale Peterson established the striking result that the cardinality of Ξ was 2^l. His ingenious proof, using the affine Weyl group, sets up a natural bijection between Ξ and the set of elements of order 2 (and the identity) in a maximal torus of a simply-connected Lie group G with Lie

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