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Selberg's Eigenvalue Conjecture and the Siegel Zeros for Hecke L-series

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The present paper is an improved and corrected version of our previous paper [B-V]. The result deals with odd eigenfunctions for all the Hecke groups $\Gamma_0(N)$. This case does not seem to be accessible to methods based on the functional equation only. An important modification is, that we have to restrict ourselves to proving the absence of eigenvalues in an interval $(\frac{1}{4} - C(N), \frac{1}{4})$, where $C(N)$ is an efficient constant, depending on N, instead of $(0, \frac{1}{4})$. In the s-plane this corresponds to the interval $(\frac{1}{2}, \frac{1}{2} + \delta(N))$, where $\delta(N) = \frac{C}{\log N}$ and C is an explicit constant independent of N. It is crucial for the proof that the interval $(1 - \delta(N), 1)$ is free from zeros of the Dirichlet series associated with the cusp form eigenfunction with eigenvalue in $(\frac{1}{2}, \frac{1}{2} + \delta(N))$. Here we apply recent remarkable results by Hoffstein and Ramakrishnan [H-R] on the absence of Siegel zeros, see also [G-H-L]. Any improvement of these results will lead to an improvement of our results on the Selberg conjecture. It is noteworthy that the method of singular perturbations as opposed to previous methods leads to absence of eigenvalues in an interval near the continuous spectrum.

We indicate the ideas of the basic steps of the proof, giving first a few definitions.

Let $\Gamma(1) = PSL(2, \mathbb{Z})$ be the modular group and

$$\Gamma(N) = \{\gamma \in \Gamma(1) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N, N \in \mathbb{Z}_+\}$$

the principal congruence subgroup of $\Gamma(1)$ of level N. By definition Γ is a congruence subgroup of $\Gamma(1)$ if there exists N with the property $\Gamma(N) \subseteq \Gamma \subseteq \Gamma(1)$. The group Γ is a cofinite, discrete subgroup of $G = PSL(2, \mathbb{R})$. G acts on the hyperbolic plane (the upper half-plane) $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, z = x + iy, y > 0$, by linear-fractional transformations, which are isometric relative to the Poincaré metric

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