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Lie-Drach-Vessiot Theory

Infinite dimensional differential Galois theory

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Introduction

Despite trials of several authors since the 19-th century, at least to our taste, infinite dimensional differential Galois theory is unfinished. We propose an infinite dimensional differential Galois theory based on a rigorous foundation. This note is prepared for the non-specialists as an introduction to our papers [U5], [U6], where interested readers can find details. After we briefly recall the history and the principle of Galois theory, we show the marvelous ideas of the classical authors on infinite dimensional differential Galois theory as well as the problems which their ideas give rise to. We can avoid all these difficulties and attach to an ordinary differential field extension L/K of finite type, or intuitively to an ordinary algebraic differential equation, a formal group **Inf-gal** of infinite dimension. **Inf-gal** is a new invariant of an ordinary algebraic differential equation. In fact, no such invariants were known. We explain an application to be expected of the invariant **Inf-gal** to the Painlevé equations in §6. A brief account on the formal group of infinite dimension and the construction of **Inf-gal** is also given.

All the rings that we consider are commutative and unitary $\mathbb{Q}$ -algebras.

§1. History

Galois (1811–32) and Abel (1802–29) invented Galois theory of algebraic equations. Their purpose was proving the impossibility of solving a general algebraic equation of degree 5 by extraction of radicals. This historical problem is the origin of Galois theory but the significance of Galois theory is prominent in later developments of number theory. We cannot speak of algebraic number theory, class field theory ... etc. without Galois theory.

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