Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 221–236

Spaces of Cauchy-Riemann Manifolds

László Lempert

§1. Introduction.

This work deals with the embeddability problem for three dimensional, compact, strongly pseudoconvex Cauchy-Riemann (CR) manifolds. Such a CR manifold is given by a compact manifold M without boundary, $\dim M = 3$; a rank two subbundle $H \subset TM$; and an endomorphism $J : H \to H$ that satisfies $J^2 = -\mathrm{id}$. (All manifolds, bundles etc. in this paper are C^∞ smooth.) Strong pseudoconvexity means that for any nonzero local section X of H the vector field $[X, JX]$ is transverse to H; or, equivalently, H defines a contact structure on M. By declaring the frame $X, JX, [JX, X]$ positively oriented, M acquires a canonical orientation.

A C^1 function $f : M \to \mathbb{C}$ is CR if it satisfies the tangential Cauchy-Riemann equations

$$Xf + iJXf = 0, \qquad X \in H. \tag{1.1}$$

A central problem of the theory is to understand how many solutions (1.1) has; in particular, if there are sufficiently many C^∞ solutions $f_1, \ldots, f_k$ to give rise to a smooth embedding $f = (f_j) : M \to \mathbb{C}^k$ into some Euclidean space. If this is so, the CR manifold (M, H, J) is called embeddable. In contrast with the higher dimensional case (see [3]) there may be very few CR functions on a three dimensional CR manifold; in fact, typically, the only CR functions are the constants, see [4,8,10,20,21].

We would like to describe the space of all (three dimensional, compact, strongly pseudoconvex) CR manifolds (M, H, J); the subspace of embeddable manifolds; and also to understand how many non isomorphic embeddable CR manifolds there are. Here two CR manifolds

---

Received September 21, 1995
Revised November 16, 1995
Research partially supported by the NSF