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Invariant Theory of the Bergman Kernel

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Dedicated to Professor M. Kuranishi on his 70th birthday

Introduction

This article is a brief report of recent developments in Fefferman's program, proposed and initiated in [F3], concerning invariant expression of the singularity of the Bergman kernel K^{B} on the diagonal of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary. It was proved by Fefferman in [F1] that

$$K^{\mathrm{B}} = \frac{\varphi^{\mathrm{B}}}{r^{n+1}} + \psi^{\mathrm{B}} \log r \quad \text{with } \varphi^{\mathrm{B}},\, \psi^{\mathrm{B}} \in C^\infty(\overline{\Omega}), \tag{0.1}$$

where $r \in C^\infty$ is a defining function of the boundary $\partial\Omega$ such that $r > 0$ in Ω and $dr \neq 0$ on $\partial\Omega$. The problem is to choose r appropriately and express φ^{B} modulo $O^{n+1}(r)$ and ψ^{B} modulo $O^\infty(r)$ invariantly in the sense of local biholomorphic geometry. This can be compared with the asymptotic expansion of the heat kernel associated with the diagonal of a compact Riemannian manifold, where the time variable corresponds to the function r in (0.1). The boundary $\partial\Omega$ is approximated at every point by a sphere (hyperquadric), and carries a differential-geometric structure, called the CR (or pseudo-conformal) structure.

Let us employ an extrinsic approach due to Chern and Moser in [CM], [M], and put the boundary $\partial\Omega$ (formally) in Moser's normal form $N(A)$ with $A = (A^\ell_{\alpha\overline{\beta}})$ given by

$$2\,\mathrm{Re}\, z_n = |z'|^2 + \sum_{|\alpha|,|\beta|\geq 2} \sum_{\ell=0}^{\infty} A^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, (\mathrm{Im}\, z_n)^\ell,$$

where $z = (z', z_n) = (z_1, \ldots, z_{n-1}, z_n) \in \mathbb{C}^n$. (For the notation z'_α and $|\alpha|$ with ordered multi-indices α, see Subsection 1.1, (B) below.)

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