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Aspects of Prescribing Ricci Curvature

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Dedicated to Professor M. Kuranishi on his 70th birthday

§1. Introduction

This article concerns two problems involving the Ricci curvature of a Riemannian metric. In each of these problems, one seeks a metric whose Ricci curvature is prescribed in advance in some manner.

Let X be a manifold of dimension $n \geq 3$, whose tangent and cotangent bundles we denote by T and T^*, respectively. By $\bigotimes^m E$, $\bigwedge^k E$ and $S^l E$, we shall mean the m-th tensor power, k-th exterior product and the l-th symmetric product of a vector bundle E over X, respectively. Under the natural identification of $\mathrm{Hom}\,(T, T^*)$ with $T^* \otimes T^*$, we can view a symmetric 2-form R on X, that is, a section of S^2T^*, as a morphism $R^\flat : T \to T^*$; we shall consider the section $\det R$ of the line bundle $\mathrm{Hom}\,(\bigwedge^n T, \bigwedge^n T^*)$ which is induced by $R^\flat$.

The first problem consists in finding a Riemannian metric with prescribed Ricci tensor. We are given a section R of S^2T^* over X and we seek a Riemannian metric g in some neighborhood of a given point $x_0 \in X$ whose Ricci tensor $\mathrm{Ric}(g)$ is equal to R throughout this neighborhood. The first definitive results concerning the problem of prescribing the Ricci tensor were obtained in [4]. There it was shown that, if $R(x_0)$ is a non-degenerate symmetric quadratic form on T_{x_0}, then a solution of this problem always exists. Examples were also given showing that, when $R(x_0)$ is degenerate, a solution may or may not exist. In the present paper, our attention focuses on the problem of solving the equation $\mathrm{Ric}(g) = R$ when R is degenerate at *every* point of X, but has constant rank.

The second problem we consider here is the prescription of the principal Ricci curvatures of a Riemannian metric (without any prescription

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