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Spherical Designs and Tensors

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§1. Introduction

The set X of the 12 vertices of a regular icosahedron on the unit sphere Ω in $\mathbf{R}^3$ provides a first example of a spherical 5-design (of strength 5). It satisfies

$$\frac{1}{n}\sum_{x\in X} h(x) = \int_\Omega h(u)\,d\sigma(u),$$

short, $\operatorname*{Ave}_X h = \operatorname*{Ave}_\Omega h$ for all polynomials h in 3 variables of degree ≤ 5 and $n = 12$. If the defining relation only refers to the homogeneous polynomials of degree q, then we use the term spherical design of index q. Thus strength q means index 1, 2, ..., and q.

The second part of the title refers to symmetric tensors, and to the desire to express symmetric polynomials as the inner products of tensors, for instance

$$\sum_{i,j,k=1}^{d} h_{ijk}a_i a_j a_k = \langle h, a\otimes a\otimes a\rangle,$$

where $a = (a_1, a_2, \ldots, a_d) \in \mathbf{R}^d =: V$. The linear space $S^q(V)$ of the symmetric q-tensors on V is spanned by the q-fold tensor powers $\otimes^q a := a\otimes a\otimes\cdots\otimes a$. This space is isomorphic to the space $\operatorname{Hom}_q(V)$ of homogeneous polynomials of degree q in d variables.

Section 2 deals with tensors in $\mathbf{R}^d$, in particular with the distribution q-tensor D, and the Sidelnikov inequality. In Section 3 this leads to the tensor-definition of spherical designs of index q and of strength q. This notion was introduced by Delsarte, Goethals and Seidel, and was further developed by Bannai. We recall that the combinatorial t-(v,k,λ) design can be phrased in analogous terms. Generalization to t-(v,K,λ)

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