Advanced Studies in Pure Mathematics 24, 1996
Progress in Algebraic Combinatorics
pp. 265–283

Incidence Matrix Diagonal Forms and Integral Hecke Algebras

Robert A. Liebler

§1. Introduction

Let $\mathcal{P}_k$ denote the set of flats in the projective geometry $PG_{n-1}(q)$ arising from the k-dimensional subspaces of an n-dimensional vector space over the Galois field $GF(q)$, q a power of the prime p. Let $\mathcal{M}_k(q)$ be incidence matrix of points $\mathcal{P}_1$ versus k-flats $\mathcal{P}_k$.

The study of $\mathbb{Z}$-span of the columns of $\mathcal{M}_k(q)$ as a submodule of $\mathbb{Z}\mathcal{P}_1$ is of interest in its own right, as a source of easily implemented codes and because it may provide a means for representing, and ultimately characterizing, associated combinatorial structures like the q-analog Johnson schemes. I also find this study particularly attractive because it provides an explicit setting to further develop the application of integral representation theory and number theory to combinatorics along the lines that have been so successful with non-abelian difference sets.

It is easy to see that the Smith normal form of $\mathcal{M}_k(q)$ has all but one of its diagonal entries a divisor of the difference : $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q - \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q$ of the q-binomial coefficients. Even for $n = 3$, every conceivable p-elementary divisor arises. Indeed, the set of lines of an affine net of degree p^k is in the kernel of the incidence map mod p^k but not mod p^{k+1}.

The first formulas for the p-rank of $\mathcal{M}_k(q)$ are due to Hamada [6] and to Smith [12] for $k = n - 1$. More recently Black and List [1] gave a generating function for the entries in a diagonal form of $\mathcal{M}_{n-1}(p)$ over $\mathbb{Z}$. Also, Lander [10, p 77] has apparently given information equivalent to a diagonal form of $\mathcal{M}_{n-1}(q)$ over $\mathbb{Z}$ in his (unpublished) Ph.D. thesis.

We show how diagonal forms for these incidence matrices arise naturally in the study of integral Hecke algebras and their geometrically significant eigenpotents. Within the $\mathbb{Z}$-module based on the chambers

Received February 7, 1995.
Revised January 15, 1996.
This research was partially supported by NSA grant 904-91-H-0048.