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Level-Rank Duality of Witten's 3-Manifold Invariants

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§1. Introduction

The main object of this paper is to establish a duality satisfied by Witten's 3-manifold invariants for $sl(n, \mathbf{C})$ at level k and those for $sl(k, \mathbf{C})$ at level n. This type of duality, which is called the level-rank duality, has been encountered in several contexts in solvable lattice models and conformal field theory — the Boltzmann weights of solvable lattice models [JMO], quantum groups at roots of unity [SA], link invariants related to Chern-Simons gauge theory [NRS], fusion algebras [KN], and the space of conformal blocks in Wess-Zumino-Witten conformal field theory [NT]. This subject has been also treated by many other authors from different viewpoints. More recently, Witten [W2] described the relationship between the fusion algebra and the quantum cohomology of the Grassmann manifold and explained the level-rank duality from this point of view. However, a precise formulation for the level-rank duality of Witten's 3-manifold invariants has not appeared in the literature, as far as the authors know.

Let M be a closed oriented 3-manifold and we denote by $Z_k(M, SU(n))$ Witten's 3-manifold invariant for $sl(n, \mathbf{C})$ at level k discovered in the seminal article [W1]. Subsequently these invariants were studied in detail in [RT] and [KM]. Our notation corresponds to $\tau_r(M)$ in [KM] with $r = k + 2$ in the case $n = 2$. To describe the duality between $Z_k(M, SU(n))$ and $Z_n(M, SU(k))$ we first factorize the invariants by the Dynkin diagram automorphism. This is the $sl(n, \mathbf{C})$ counterpart of the symmetry principle discovered in [KM] for $sl(2, \mathbf{C})$. In this paper we assume that the integers n and k are relatively prime. Let us suppose that M is obtained by the Dehn surgery on a framed link L in S^3. We recall that the invariant $Z_k(M, SU(n))$ is written as a weighted sum of link invariants obtained by associating with each component of the link

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