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An $L^{q,r}$-Theory for Nonlinear Schrödinger Equations

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§1. Introduction

Consider the nonlinear Schrödinger equation:

$$\text{(NLS)} \qquad \partial_t u = i(\Delta u - F(u)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^m,$$

where $F(u) = F \circ u$ is, for example, a Nemyckii operator defined by a function $F : \mathbb{C} \to \mathbb{C}$. There is an extensive literature on this problem, but it seems that all existing work assumes that either the initial value $\phi = u(0) = u(0, \cdot)$ or the limit $\phi_\pm = \lim_{t\to\pm\infty} e^{-it\Delta}u(t)$ is in L^2. The present paper is an attempt to solve (NLS) with the data in a larger class of functions.

As in most of the work on (NLS), we convert (NLS) into integral equations such as

$$\text{(INT)} \qquad u = \Phi u \equiv u_0 - iGF(u), \quad \text{or} \quad u = \Phi_\pm u \equiv u_\pm - iG_\pm F(u).$$

Here u_0 or $u_\pm$ is a *free wave* (solution of the free Schrödinger equation $\partial_t u = i\Delta u$), and G or $G_\pm$ is an integral operator defined by

$$(1.1) \qquad \begin{aligned} Gf(t) &= \int_0^t U(t-s) f(s)\, ds, \\ G_\pm f(t) &= \int_{\pm\infty}^t U(t-s) f(s)\, ds, \quad U(t) = e^{it\Delta}. \end{aligned}$$

The free term u_0 in (INT) is usually related to the initial value $u(0) = \phi$ by

$$(1.2) \qquad u_0 = \Gamma\phi, \quad \Gamma\phi(t) = U(t)\phi,$$

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