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Differential Systems Associated with Simple Graded Lie Algebras

Keizo Yamaguchi

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

§0. Introduction

This is a survey paper on differential systems associated with simple graded Lie algebras. By a differential system (M, D), we mean a pfaffian system D (or a distribution in Chevalley's sense) on a manifold M, that is, D is a subbundle of the tangent bundle $T(M)$ of M. Our primary subject will be the Lie algebra (sheaf) $\mathcal{A}(M, D)$ of all infinitesimal automorphisms of (M, D).

Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{R}$ of real numbers. A gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \qquad \text{for } p,\ q \in \mathbb{Z}.$$

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. We denote by G the adjoint group of $\mathfrak{g}$ and let G' be the normalizer of $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ in G;

$$G' = \{\, \sigma \in G \mid \sigma(\mathfrak{g}') = \mathfrak{g}' \,\}.$$

We consider the homogeneous space $M_{\mathfrak{g}} = G/G'$, which is a real or complex manifold (R-space) depending on whether the complexification $\mathbb{C}\mathfrak{g}$ of $\mathfrak{g}$ is simple or $\mathfrak{g}$ is complex simple (see Proposition 3.3 in §3.2 and §4.1). By identifying $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on G, the G'-invariant subspace $\mathfrak{f}^{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}'$ induces a G-invariant differential system $D_{\mathfrak{g}}$ on $M_{\mathfrak{g}}$, which is a holomorphic differential system when $\mathfrak{g}$ is complex simple. $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is called the standard differential system of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (§4.1).

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