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Diameter and Area Estimates for S^2 and P^2 with Nonnegatively Curved Metrics

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§0. Introduction

We consider the quantity

$$F(M) := \frac{\mathrm{Vol}(M)}{\mathrm{Diam}(M)^n}$$

for any closed Riemannian n-manifold M, which is a homothety invariant, where Vol and Diam denote the volume and the diameter respectively. If the Ricci curvature of M is nonnegative everywhere, Bishop's volume comparison theorem implies that $F(M) < \pi$. A.D. Alexandrov conjectured in [A, p.417] (see also [BZ, p.42]) that for any nonnegatively curved metric g on the 2-sphere S^2,

$$F(S^2, g) \leq \frac{\pi}{2},$$

and the equality holds only if g is homothetic to the metric of the double of the Euclidean unit disk $\bar{B}(1) := \{ x \in \mathbf{R}^2 \mid d(x, o) \leq 1 \}$, which is a singular metric of nonnegative Toponogov curvature. Note that Alexandrov deals a class of surfaces containing such a singular space, namely surfaces of bounded curvature in the sense of [AZ]. The volume and the diameter of any such singular surface can be approximated by those of Riemannian 2-manifolds, and thus it suffices to consider only regular metrics.

Alexandrov's conjecture has not been proved as of now. Concerning this, there are two known results as follows.

Theorem (Sakai, [S]). *For any nonnegatively curved Riemannian metric g on the 2-sphere S^2,*

$$F(S^2, g) < 0.985\pi.$$

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